Orthogonal Matrices & Symmetric Matrices

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Outline

Orthogonal Matrices

- Reference: Chapter 7.5

Symmetric Matrices

- Reference: Chapter 7.6

Norm-preserving

- A linear operator is norm-preserving if

$$\|T(u)\| = \|u\| \qquad \text{For all u}$$

Example: linear operator *T* on $\mathscr{R}^2$ that rotates a vector by θ.

$\Rightarrow$ Is *T* norm-preserving?

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example: linear operator *T* is refection

$\Rightarrow$ Is *T* norm-preserving?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Norm-preserving

- A linear operator is norm-preserving if

$$\|T(u)\| = \|u\| \qquad \text{For all u}$$

Example: linear operator T is projection $\qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ Is T norm-preserving?

Example: linear operator U on $\mathcal{R}^n$ that has an eigenvalue $\lambda \neq \pm 1$.

Orthogonal Matrix

- An nxn matrix Q is called an orthogonal matrix (or simply orthogonal) if the columns of Q form an orthonormal basis for R^n
- Orthogonal operator: standard matrix is an orthogonal matrix.

$$A_\theta = \begin{array}{c} \begin{array}{cc} \text{unit} & \text{unit} \end{array} \\ \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{orthogonal}} \end{array}$$ is an orthogonal matrix.

Norm-preserving

- Necessary conditions:

Norm-preserving $\Longrightarrow$ Orthogonal Matrix

Norm-preserving $\overset{???}{\Longleftarrow}$ Orthogonal Matrix

Linear operator Q is norm-preserving

$\Rightarrow$ $\|\mathbf{q}_j\| = 1$ $\qquad$ $\|\mathbf{q}_j\| = \|Q\mathbf{e}_j\| = \|\mathbf{e}_j\|$

$\Rightarrow$ $\mathbf{q}_i$ and $\mathbf{q}_j$ are orthogonal $\qquad$ 畢式定理

$$\|\mathbf{q}_i + \mathbf{q}_j\|^2 = \|Q\mathbf{e}_i + Q\mathbf{e}_j\|^2 = \|Q(\mathbf{e}_i + \mathbf{e}_j)\|^2 = \|\mathbf{e}_i + \mathbf{e}_j\|^2 = 2 = \|\mathbf{q}_i\|^2 + \|\mathbf{q}_j\|^2$$

Orthogonal Matrix

Those properties are used to check orthogonal matrix.

- Q is an orthogonal matrix
- $QQ^T = I_n$
- Q is invertible, and $Q^{-1} = Q^T$
- $Qu \cdot Qv = u \cdot v$ for any u and v
- $\|Qu\| = \|u\|$ for any u

Simple inverse

Q preserves dot projects

Q preserves norms

Norm-preserving $\rightleftarrows$ Orthogonal Matrix

Orthogonal Matrix

- Let P and Q be n x n orthogonal matrices
  - $detQ = \pm 1$
  - PQ is an orthogonal matrix
  - Q^{-1} is an orthogonal matrix
  - Q^T is an orthogonal matrix

Check by $(PQ)^{-1} = (PQ)^T$

Check by $(Q^{-1})^{-1} = (Q^{-1})^T$

Proof

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$$

Rows and columns

Orthogonal Operator

- Applying the properties of orthogonal matrices on orthogonal operators
- T is an orthogonal operator
  - $T(u) \cdot T(v) = u \cdot v$ for all u and v
  - $\|T(u)\| = \|u\|$ for all u

Preserves dot product

Preserves norms

- T and U are orthogonal operators, then TU and T^{-1} are orthogonal operators.

Example: Find an orthogonal operator T on $\mathcal{R}^3$ such that

$$T\left(\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Norm-preserving

$$v = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \qquad Av = e_2 \qquad v = A^{-1}e_2$$

Find A^{-1} first

Because $A^{-1} = A^T$

$$A^{-1} = \begin{bmatrix} * & 1/\sqrt{2} & * \\ * & 0 & * \\ * & 1/\sqrt{2} & * \end{bmatrix}$$

Also orthogonal

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$A = (A^{-1})^T$$

Conclusion

- Orthogonal Matrix (Operator)
  - Columns and rows are orthogonal unit vectors
  - Preserving norms, dot products
  - Its inverse is equal its transpose

Outline

Orthogonal Matrices

- Reference: Chapter 7.5

Symmetric Matrices

- Reference: Chapter 7.6

Eigenvalues are real

- The eigenvalues for symmetric matrices are always real.

Consider 2 x 2 symmetric matrices

How about more general cases?

實係數多項式虛根共軛

$$A = A^T = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in R^{2\times 2}$$

$$det(A - tI_2) = t^2 - (a+c)t + ac - b^2$$

$$\text{Since } (a+c)^2 - 4(ac - b^2) = (a-c)^2 + 4b^2 \geq 0$$

The symmetric matrices always have real eigenvalues.

Orthogonal Eigenvectors

A is symmetric

$det(A - tI_n)$ Factorization

$$= (t-\lambda_1)^{\underline{m_1}}(t-\lambda_2)^{\underline{m_2}} \dots (t-\lambda_k)^{\underline{m_k}}(\dots\dots)$$

Eigenvalue: λ_1 λ_2 …… λ_k

Eigenspace: (dimension) $d_1 \le m_1$ $d_2 \le m_2$ …… $d_k \le m_k$

Independent → orthogonal

Orthogonal Eigenvectors

- A is symmetric.
- If u and v are eigenvectors corresponding to eigenvalues λ and μ ($\lambda \neq \mu$)

$\Rightarrow$ u and v are orthogonal.

$$A\mathbf{u} \cdot \mathbf{v} = \lambda \mathbf{u} \cdot \mathbf{v}$$

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T \mathbf{v}$$

$$= \mathbf{u} \cdot \mu \mathbf{v}$$

Diagonalization

A is symmetric

P560 ?

$A = PDP^T$

$P^T AP = D$

P is an orthogonal matrix

D is a diagonal matrix

: simple

$P^T AP = D$

$P^{-1} AP = D$

$A = PDP^{-1}$

$A = PDP^T$

Diagonalization

P consists of eigenvectors , D are eigenvalues

Diagonalization

- Example

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

$\mathrm{A} = P\mathrm{D}\mathrm{P}^{-1}$ ⟹ $\mathrm{A} = P\mathrm{D}\mathrm{P}^{T}$, $\mathrm{P}^{T}\mathrm{A}P = \mathrm{D}$

A has eigenvalues $\lambda_1 = 6$ and $\lambda_2 = 1$,

with corresponding eigenspaces $\mathcal{E}_1 = \mathrm{Span}\{[\,-1\ \ 2\,]^T\}$ and $\mathcal{E}_2 = \mathrm{Span}\{[\,2\ \ 1\,]^T\}$

orthogonal

$\Rightarrow \mathcal{B}_1 = \{[\,-1\ \ 2\,]^T/\sqrt{5}\}$ and $\mathcal{B}_2 = \{[\,2\ \ 1\,]^T/\sqrt{5}\}$

$$P = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example of Diagonalization of Symmetric Matrix

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$A = PDP^{-1}$ → $A = PDP^{T}$

P is an orthogonal matrix

$\lambda_1 = 2$

Intendent

Eigenspace: $Span\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ $\xrightarrow[\text{normalization}]{\text{Gram-Schmidt}}$ $Span\left\{\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}\right\}$

$\lambda_2 = 8$

Not orthogonal

Eigenspace: $Span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ $\xrightarrow[\text{normalization}]{}$ $Span\left\{\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\right\}$

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Diagonalization

P is an orthogonal matrix

A is symmetric $\Leftrightarrow$ $P^T A P = D$

$A = P D P^T$

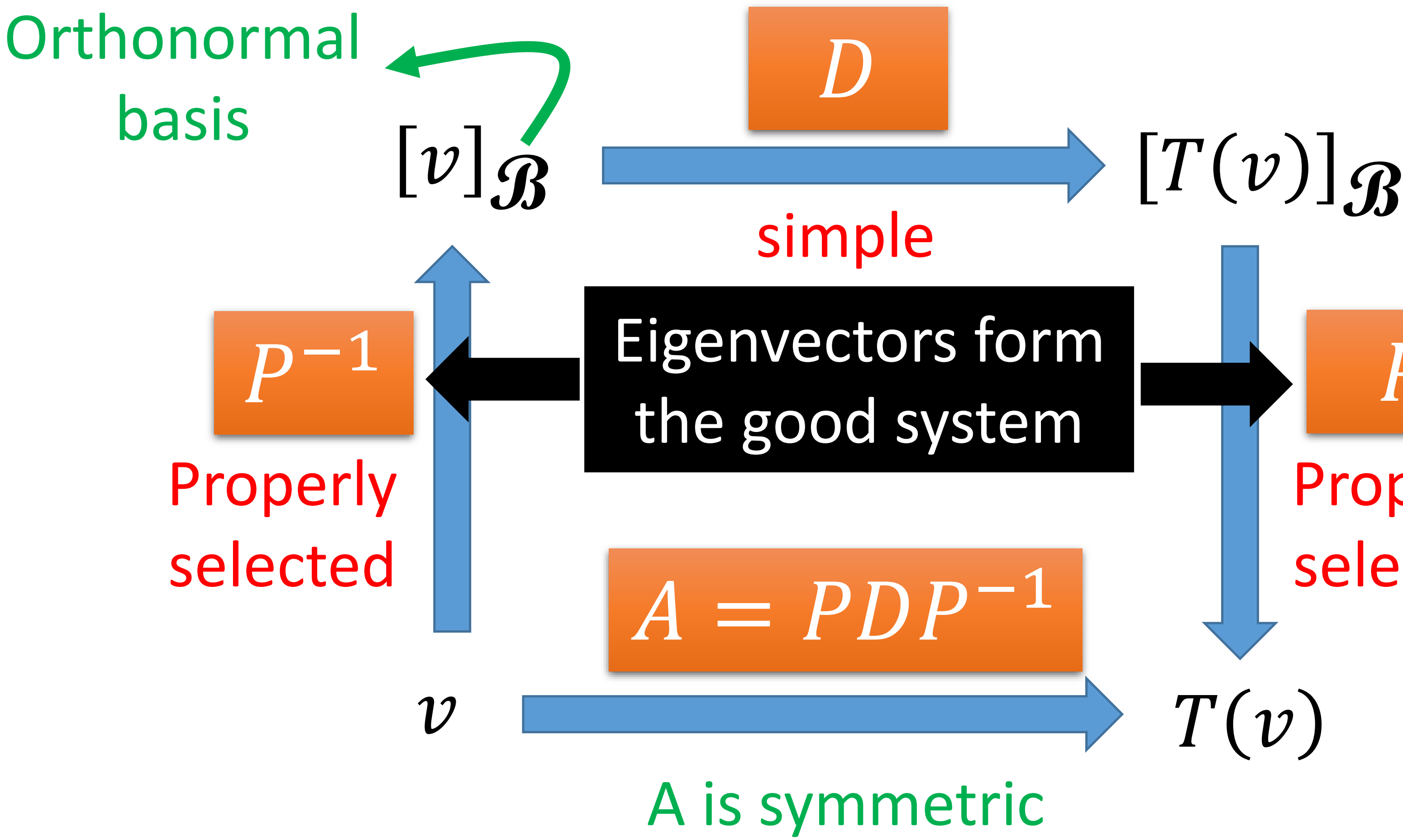
P consists of eigenvectors , D are eigenvalues

Finding an orthonormal basis consisting of eigenvectors of A

Diagonalization of Symmetric Matrix

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$$u \cdot v_1 \qquad u \cdot v_2 \qquad u \cdot v_k$$



Spectral Decomposition

Orthonormal basis

$A = PDP^T$ Let $P = [\ \mathbf{u}_1\ \ \mathbf{u}_2\ \ \cdots\ \ \mathbf{u}_n\ ]$ and $D = \text{diag}[\ \lambda_1\ \ \lambda_2\ \ \cdots\ \ \lambda_n\ ]$.

$$= P[\ \lambda_1\mathbf{e}_1\ \ \lambda_2\mathbf{e}_2\ \ \cdots\ \ \lambda_n\mathbf{e}_n\ ]P^T$$

$$= [\ \lambda_1 P\mathbf{e}_1\ \ \lambda_2 P\mathbf{e}_2\ \ \cdots\ \ \lambda_n P\mathbf{e}_n\ ]P^T$$

$$= [\ \lambda_1\mathbf{u}_1\ \ \lambda_2\mathbf{u}_2\ \ \cdots\ \ \lambda_n\mathbf{u}_n\ ]\begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

P_1 P_2 P_n

$$= \lambda_1 \mathrm{P}_1 + \lambda_2 \mathrm{P}_2 + \cdots + \lambda_n \mathrm{P}_n$$

P_i are symmetric

Spectral Decomposition

Orthonormal basis

$A = PDP^T$ Let $P = [\ \mathbf{u}_1\ \ \mathbf{u}_2\ \ \cdots\ \ \mathbf{u}_n\ ]$ and $D = \mathrm{diag}[\ \lambda_1\ \ \lambda_2\ \ \cdots\ \ \lambda_n\ ]$.

$$= \lambda_1 \mathrm{P}_1 + \lambda_2 \mathrm{P}_2 + \cdots + \lambda_n \mathrm{P}_n$$

$$\operatorname{rank} P_i = \operatorname{rank} \mathbf{u}_i \mathbf{u}_i^T = 1.$$

$$P_i P_i = \mathbf{u}_i \underline{\mathbf{u}_i^T \mathbf{u}_i} \mathbf{u}_i^T = \mathbf{u}_i \mathbf{u}_i^T$$

$$P_i P_j = \mathbf{u}_i \underline{\mathbf{u}_i^T \mathbf{u}_j} \mathbf{u}_j^T = O$$

$$P_i \mathbf{u}_i$$

$$P_i \mathbf{u}_j$$

Spectral Decomposition

- Example

$$A = \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}$$ Find spectrum decomposition.

Eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -5$.

$$P_1 = u_1 u_1^T$$

An orthonormal basis consisting of eigenvectors of *A* is

$$B = \left\{ \underbrace{\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}}_{u_2} \right\}$$

$$P_2 = u_2 u_2^T$$

$$A = \lambda_1 P_1 + \lambda_2 P_2$$

Conclusion

- Any symmetric matrix
  - has only real eigenvalues
  - has orthogonal eigenvectors.
  - is always diagonalizable

A is symmetric $\Leftrightarrow$ $P^T A P = D$ $\quad$ $A = PDP^T$

P is an orthogonal matrix

Appendix

Diagonalization

- By induction on n.
- $n = 1$ is obvious.
- Assume it holds for $n \geq 1$, and consider $A \in \mathscr{R}^{(n+1)\times(n+1)}$.
- A has an eigenvector $\mathbf{b}_1 \in \mathscr{R}^{n+1}$ corresponding to a real eigenvalue λ, so $\exists$ an orthonormal basis $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n+1}\}$
  - by the **Extension Theorem** and Gram-Schmidt Process.

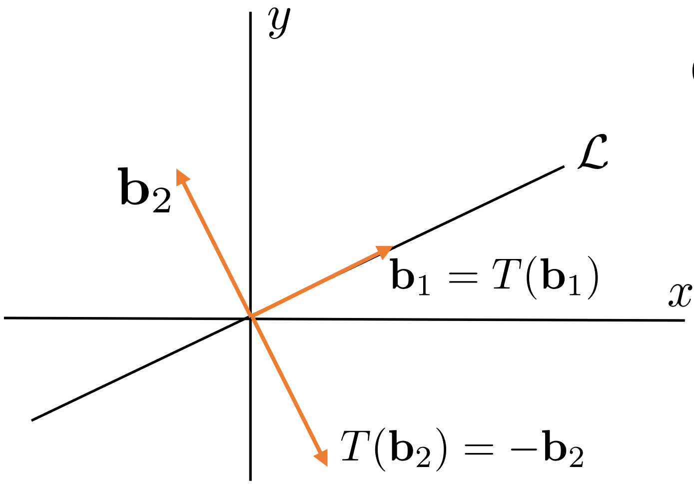
$$B^T AB = \begin{bmatrix} \mathbf{b}_1^T \\ \mathbf{b}_2^T \\ \vdots \\ \mathbf{b}_{n+1}^T \end{bmatrix} \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_{n+1} \end{bmatrix} = \left[\begin{array}{c|ccc} \mathbf{b}_1^T A\mathbf{b}_1 & \mathbf{b}_1^T A\mathbf{b}_2 & \cdots & \mathbf{b}_1^T A\mathbf{b}_{n+1} \\ \hline \mathbf{b}_2^T A\mathbf{b}_1 & \mathbf{b}_2^T A\mathbf{b}_2 & \cdots & \mathbf{b}_2^T A\mathbf{b}_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_{n+1}^T A\mathbf{b}_1 & \mathbf{b}_{n+1}^T A\mathbf{b}_2 & \cdots & \mathbf{b}_{n+1}^T A\mathbf{b}_{n+1} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \lambda & \mathbf{0}^T \\ \hline \mathbf{0} & S \end{array}\right], \text{since } \mathbf{b}_1^T A\mathbf{b}_1 = \lambda \mathbf{b}_1^T \mathbf{b}_1 = \lambda \text{ and } \mathbf{b}_j^T A\mathbf{b}_1 = \mathbf{b}_1^T A\mathbf{b}_j = 0 \; \forall j \neq 1.$$

$S = S^T \in \mathcal{R}^{n\times n} \Rightarrow \exists$ an orthogonal $C \in \mathcal{R}^{n\times n}$ and a diagonal $L \in \mathcal{R}^{n\times n}$ such that $C^T SC = L$ by the induction hypothesis.

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & C^T \end{bmatrix}}_{\text{orthogonal } P^T} B^T AB \underbrace{\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & C \end{bmatrix}}_{\text{orthogonal } P} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & C^T \end{bmatrix} \begin{bmatrix} \lambda & \mathbf{0}^T \\ \mathbf{0} & S \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & C \end{bmatrix} = \begin{bmatrix} \lambda & \mathbf{0}^T \\ \mathbf{0} & C^T SC \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda & \mathbf{0}^T \\ \mathbf{0} & L \end{bmatrix}}_{\text{diagonal } D}$$

Example: reflection operator T about a line $\mathscr{L}$ passing the origin.



Question: Is T an orthogonal operator?

(An easier) Question:
Is T orthogonal if $\mathscr{L}$ is the x-axis?

$\mathbf{b}_1$ is a unit vector along $\mathscr{L}$.
$\mathbf{b}_2$ is a unit vector perpendicular to $\mathscr{L}$.
$P = [\ \mathbf{b}_1\ \ \mathbf{b}_2\ ]$ is an orthogonal matrix.
$\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is an orthonormal basis of $\mathscr{R}^2$.
$[T]_{\mathscr{B}} = \text{diag}[1\ -1]$ is an orthogonal matrix.

Let the standard matrix of T be Q. Then $[T]_{\mathscr{B}} = P^{-1}QP$, or $Q = P[T]_{\mathscr{B}}P^{-1} \Rightarrow Q$ is an orthogonal matrix. $\Rightarrow T$ is an orthogonal operator.